

# A Canonical Form of the Metric in General Relativity

Ying-Qiu Gu\*

*School of Mathematical Science, Fudan University, Shanghai 200433, China*

If there is a null gradient field in a 1+3 dimensional Lorentzian manifold, we can establish a kind of light-cone coordinate system for the manifold from the null gradient field. In such coordinate system, the metric takes a wonderful canonical form, which is much helpful for resolving the Einstein's field equation. In this paper, we show how to construct the new coordinate system, and then explain their geometrical and physical meanings via examples. In the light-cone coordinate system, the complicated Einstein's field equation could be greatly simplified. This coordinate system might be also helpful to understand the propagation of the gravitational wave.

Keywords: *light-cone coordinate, canonical metric, null gradient field, Einstein equation*

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## I. INTRODUCTION

A good choice for the coordinate system of the space-time manifold is much helpful to solve the Einstein's equation. The traditional choices are the Gaussian normal coordinates system and the harmonic coordinates system[1]. The former gives the following metric locally

$$G_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & g_{11} & g_{12} & g_{13} \\ 0 & g_{21} & g_{22} & g_{23} \\ 0 & g_{31} & g_{32} & g_{33} \end{pmatrix}, \quad (1.1)$$

and the latter satisfies the following coordinate condition

$$\Gamma^\mu = g^{\alpha\beta} \Gamma_{\alpha\beta}^\mu = 0. \quad (1.2)$$

However (1.1) and (1.2) are only convenient for theoretical analysis rather than for practical solution of the Einstein's field equation. The conventional method to get the exact solution of Einstein's

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\*Electronic address: [yqgu@fudan.edu.cn](mailto:yqgu@fudan.edu.cn)

equation is usually to analysis the symmetry of the space-time. Almost all the well known solutions such as the Friedmann-Robertson-Walker metric, Bianchi universe, Lemaitre-de Sitter universe, as well as Schwarzschild metric and Kerr metric, Taub-NUT solution[1, 2, 3, 4], are all related with some special symmetry of the space-time.

In this paper, we present a new kind coordinate system, namely the light-cone coordinate system and corresponding canonical metric form. The light-cone coordinate system is induced from a global set of null geodesic. The geometrical and physical meanings are discussed via examples. In such coordinate system, the complicated Einstein's field equation could be greatly simplified, and the exact solutions could be more easily obtained[5].

## II. THE LIGHT-ONE COORDINATE SYSTEM

**Lemma 1.** *For the following canonical metric of the 1+3 dimensional space-time with coordinate system  $(t, z, x, y)$*

$$g_{\mu\nu} = \begin{pmatrix} u^2 & v & p & q \\ v & 0 & 0 & 0 \\ p & 0 & -a^2 & 0 \\ q & 0 & 0 & -b^2 \end{pmatrix}, \quad (2.1)$$

*whose form is invariant under the following coordinate transformation*

$$t = f_0(t'), \quad z = f_1(t', z'), \quad x = f_2(t', x'), \quad y = f_3(t', y'), \quad (2.2)$$

*where  $f_\mu$  are any given smooth functions.*

**Proof.** Lemma 1 can be checked by direct calculation. Computing the line element of the space-time, we have

$$\begin{aligned} ds^2 &= u^2 dt^2 + 2dt(vdz + pdx + qdy) - a^2 dx^2 - b^2 dy^2, \\ &= U^2 dt'^2 + 2dt'(Vdz' + Pdx' + Qdy') - A^2 dx'^2 - B^2 dy'^2, \end{aligned} \quad (2.3)$$

where

$$\begin{aligned} U &= \sqrt{\left(u \frac{df_0}{dt'}\right)^2 + 2 \frac{df_0}{dt'} \left(v \frac{\partial f_1}{\partial t'} + p \frac{\partial f_2}{\partial t'} + q \frac{\partial f_3}{\partial t'}\right) - \left(a \frac{\partial f_2}{\partial t'}\right)^2 - \left(b \frac{\partial f_3}{\partial t'}\right)^2}, \\ P &= \frac{\partial f_2}{\partial x'} \left(p \frac{df_0}{dt'} - a^2 \frac{\partial f_2}{\partial t'}\right), \quad Q = \frac{\partial f_3}{\partial y'} \left(q \frac{df_0}{dt'} - b^2 \frac{\partial f_3}{\partial t'}\right), \\ V &= v \frac{df_0}{dt'} \frac{\partial f_1}{\partial z'}, \quad A = a \frac{\partial f_2}{\partial x'}, \quad B = b \frac{\partial f_3}{\partial y'}. \end{aligned}$$

The proof is finished.

Obviously, as a supplement of Lemma 1, we have

**Remark 2.** *Except for the transformation (2.2), the canonical metric (2.1) also keeps the form under orthogonal coordinate transformations between  $(x, y)$  if  $a$  and  $b$  take the conformal forms*

$$a^2 dx^2 + b^2 dy^2 = \tilde{g}^2(t, x, y) \left( \tilde{a}^2(x, y) dx^2 + \tilde{b}^2(x, y) dy^2 \right). \quad (2.4)$$

Lemma 1 and 2 reveal the uncertainty of the coordinate system corresponding to the metric (2.1). They are helpful for the understanding the construction of the light-cone coordinate system.

**Theorem 3.** *In the 1 + 3 dimensional Lorentzian manifold  $(M, g)$  with coordinates  $\xi^\mu$ , where  $\mu \in \{0, 1, 2, 3\}$  and  $\xi^0$  corresponds to the time-like coordinate, the metric  $g_{\mu\nu}$  can be converted into the canonical form (2.1) via a coordinate transformation, if and only if there is a smooth null vector field  $V^\mu \partial_\mu$  on  $M$*

$$g_{\mu\nu} V^\mu V^\nu = 0, \quad (2.5)$$

and the 1-form

$$\omega = g_{\mu\nu} V^\mu d\xi^\nu \quad (2.6)$$

is integrable.

**Proof.** For the sufficient part, define

$$dt = K g_{\mu\nu} V^\mu d\xi^\nu, \quad (2.7)$$

where  $K$  is a factor to make the 1-form  $\omega$  become an exact differential form, so the integrable condition is equivalent to the exact differential form due to (2.5). The factor  $K$  include a class of functions, but it should satisfy

$$\frac{\partial t}{\partial \xi^0} = K g_{\mu 0} V^\mu > 0, \quad (\forall \xi^0) \quad (2.8)$$

to get a 1-1 correspondence of coordinate transformation. Then we have a regular coordinate transformation

$$t = t(\xi^\mu). \quad (2.9)$$

Along any null geodesic with tangent vector  $V^\mu = \frac{d\xi^\mu(\tau)}{d\tau}$ , where  $\tau$  is the parameter of the geodesic, we have

$$dt(\tau) = g_{\mu\nu} V^\mu \frac{d\xi^\nu(\tau)}{d\tau} d\tau = g_{\mu\nu} V^\mu V^\nu d\tau = 0. \quad (2.10)$$

So for any given constant  $t_0$ , the hypersurface  $t(\xi^\mu) = t_0$  is a set of propagating light wave fronts orthogonal to  $V^\mu$ . The 2-dimensional surface  $t(\xi^m)|_{\xi^0=const.} = t_0, (m = 1, 2, 3)$  corresponds to a 2-dimensional surface  $S$ . The initial surface is  $S_0 = \{\xi^\mu \mid t(\xi^m)|_{\xi^0=C_0} = t_0\}$ , where  $C_0$  is a given constant. By the definition,  $S$  is orthogonal to null geodesics generated by  $V^\mu$ , and different  $t_0$  or  $S_0$  corresponds to different new time coordinate  $t$ .

Now we construct the coordinate  $z$  as

$$z = z(\xi^\mu). \quad (2.11)$$

We have

$$dz = \frac{\partial z}{\partial \xi^\mu} d\xi^\mu. \quad (2.12)$$

Taking the trajectories of the null geodesic, namely the light rays, as the  $z$  coordinate lines, then along these  $z$  lines we have  $d\xi^\mu = V^\mu d\tau$ . Substituting it into (2.12) we get

$$V^\mu \partial_\mu z = \frac{dz}{d\tau} \stackrel{\text{def}}{=} f, \quad (2.13)$$

where  $f > 0$  is any given smooth function, which acts as the scale of  $z$  coordinate. Solving (2.13) with boundary condition on surface  $S_0$

$$z|_{S_0} = z_0, \quad (2.14)$$

where  $z_0$  is a given constant, we obtain the coordinate transformation (2.11). The propagating distance of the propagating light wave front  $S_0 \rightarrow S$  defines the new coordinate  $z$ , so we can denote the propagating surface as  $S(t_0, z_0) \rightarrow S(t_0, z)$  for clearness.

The option of  $f$  is quite arbitrary. According to Lemma 1, we can choose any smooth function  $f(t(\xi^\mu), z) > 0$ , which has not influence on the form of the canonical metric (2.1). For the convenience of resolving (2.13), we can take  $f = 1$  or  $f = z$  or some factors of vector  $V^\mu$  to make the equation simplified.

For the 2-dimensional surface  $S(t_0, z_0)$ , not loss generality, we can assume the parameter coordinates  $(x, y)$  are orthogonal grid. Otherwise, we can take the 2 principal curves of the surface as coordinate lines  $(x, y)$  to get orthogonal coordinates. If we set each null geodesic with unique parameter coordinate  $(x, y)$ , then the coordinates  $(x, y)$  become global coordinates. Since  $z$  coordinate corresponds to an equidistant translation of the 2-dimensional surface  $S(t_0, z_0) \rightarrow S(t_0, z)$ , the  $(x, y)$  grid on the surface  $S(t_0, z)$  along the geodesic keeps orthogonal, and then all space-like coordinate bases  $(\partial_x, \partial_y, \partial_z)$  also keep orthogonal. So the spatial coordinates  $(z, x, y)$  form an

global orthogonal coordinate grid, and then the metric in new coordinate system  $(t, z, x, y)$  takes the following form

$$g_{\mu\nu} = \begin{pmatrix} u^2 & v & p & q \\ v & -w^2 & 0 & 0 \\ p & 0 & -a^2 & 0 \\ q & 0 & 0 & -b^2 \end{pmatrix}. \quad (2.15)$$

For light travels along the  $z$  lines, we have  $dx = dy = 0$ , so the interval becomes

$$0 = ds^2 = u^2 dt^2 - w^2 dz^2 + 2v dt dz. \quad (2.16)$$

By the definition of  $t$  in (2.10), we have  $dt = 0$  for the same propagating light wave front  $S(t_0, z_0) \rightarrow S(t_0, z)$ . Since  $dz \neq 0$ , then we get  $w = 0$  from (2.16). Considering the arbitrary of  $(t, z, x, y)$ , we have

$$w \equiv 0, \quad (2.17)$$

and then the new metric (2.15) becomes the canonical form (2.1).

For necessary part, solving the null geodesic equations along the  $z$  coordinate line in the space-time with the canonical metric (2.1), we get

$$\frac{d^2 z}{d\tau^2} = -\frac{\partial_z v}{v} \left( \frac{dz}{d\tau} \right)^2, \quad \frac{dt}{d\tau} = \frac{dx}{d\tau} = \frac{dy}{d\tau} = 0, \quad (2.18)$$

or the null vector field

$$V^\mu = \left( 0, \frac{\kappa}{v}, 0, 0 \right), \quad (2.19)$$

where  $\kappa$  is a constant. The 1-form (2.6) becomes

$$\omega = g_{rt} V^r dt = \kappa dt, \quad (2.20)$$

which is an exact differential form. The proof is finished.

From the above proof, the new coordinate system  $(t, z, x, y)$  is induced from a global null geodesic set, so it can be called the ‘light-cone coordinate system’. In the light-cone coordinate system, the structure of the space-time becomes quite simple, and the exact solutions to the Einstein’s field equation can be much easily obtained[4, 5, 6].

**Remark 4.** *The integrable condition of  $\omega$  in theorem 3 is equivalent to that  $\omega$  is an exact differential form, because the null vector field  $V^\mu$  can multiply an arbitrary factor function  $K(\xi^\mu)$*

due to (2.5). Then the metric can be transformed into the canonical form, if and only if the covariant speed  $V_\mu$  is a null gradient field, i.e., there is a scalar field  $T(\xi^\mu)$ , such that

$$V_\mu = \partial_\mu T. \quad (2.21)$$

**Remark 5.** For the coordinate functions

$$t = t(\xi^\mu), \quad x = x(\xi^\mu), \quad y = y(\xi^\mu) \quad (2.22)$$

defined in the theorem 3, they all satisfy the following partial differential equation

$$V^\mu \partial_\mu F(\xi^\nu) = 0. \quad (2.23)$$

**Proof.** For the function  $t(\xi^\mu)$ , the directional derivative along the  $z$  coordinate lines is just  $V^\mu$ , and then we obtain

$$V^\mu \partial_\mu t(\xi^\nu) = g_{\mu\nu} V^\mu V^\nu = 0. \quad (2.24)$$

So  $t = t(\xi^\mu)$  satisfies (2.23).

For the coordinate function  $x$ , along  $z$  coordinate lines we have

$$dx = \partial_\mu x d\xi^\mu = V^\mu \partial_\mu x d\tau = 0, \quad (2.25)$$

that is  $x = x(\xi^\mu)$  satisfies (2.23). Similarly, we can check  $y = y(\xi^\mu)$  also satisfies (2.23). The proof is finished.

The Eqs.(2.7), (2.13) and (2.23) form the basic differential equation to determine the canonical coordinate system  $(t, x, y, z)$ . The above procedure shows the physical and geometrical meanings of the light-cone coordinates  $(t, z, x, y)$  and the corresponding metric (2.1).

### III. EXAMPLES OF COORDINATE TRANSFORMATION

Some concepts of the light-cone coordinate system in a 1+3 dimensional Lorentzian manifold is not intuitive. In what follows, we take the Schwarzschild metric and the Kerr-like metric as examples to explain their geometrical meaning and show how to construct the coordinates.

For the Schwarzschild metric

$$g_{\mu\nu} = \text{diag} \left[ 1 - \frac{2m}{r}, -\left(1 - \frac{2m}{r}\right)^{-1}, -r^2, -r^2 \sin^2 \theta \right], \quad (r > 2m) \quad (3.1)$$

with the coordinate system  $(t, r, \theta, \phi)$ , the radial null geodesic satisfies

$$g_{00}\dot{t}^2 - g_{11}\dot{r}^2 = \left(1 - \frac{2m}{r}\right)\dot{t}^2 - \left(1 - \frac{2m}{r}\right)^{-1}\dot{r}^2 = 0, \quad (3.2)$$

where  $\dot{t} = \frac{dt}{d\tau}$  and  $\dot{r} = \frac{dr}{d\tau}$ . Taking the null vector orthogonal to the 2-dimensional surface  $(\theta, \phi)$  as follows

$$V^\mu = \left( \left(1 - \frac{2m}{r}\right)^{-1}, \pm 1, 0, 0 \right), \quad (3.3)$$

it is easy to check that the corresponding 1-form (2.6) is an exact differential form. The initial light wave front  $S(t_0, r_0)$  is simply a sphere with radius  $r > 2m$ .  $V^r = 1$  corresponds to the outward light rays and  $V^r = -1$  corresponds to the inward light rays. In what follows we only calculate the case of  $V^r = 1$ .

By (2.7), we get the coordinate function  $\tilde{t}$

$$\begin{aligned} \tilde{t} &= \int g_{\mu\nu} V^\mu d\xi^\nu = \int \left( dt - \left(1 - \frac{2m}{r}\right)^{-1} dr \right) \\ &= t - r - 2m \ln(r - 2m) + t_0 \end{aligned} \quad (3.4)$$

By (2.23), for  $x, y$  we have

$$V^\mu \partial_\mu F = \left(1 - \frac{2m}{r}\right)^{-1} \partial_t F + \partial_r F = 0. \quad (3.5)$$

The general solution is given by

$$F = H(\tilde{t}, \theta, \phi), \quad (3.6)$$

where  $H(\tilde{t}, \theta, \phi)$  is arbitrary smooth function. By Lemma 1 and the orthogonal condition, we can choose

$$x = \theta, \quad y = \phi. \quad (3.7)$$

By (2.13), we have

$$V^\mu \partial_\mu z = \left(1 - \frac{2m}{r}\right)^{-1} \partial_t z + \partial_r z = \left(1 - \frac{2m}{r}\right)^{-1} f. \quad (3.8)$$

For (3.8), we get typical solutions independent of  $(x, y)$

$$z = \begin{cases} r + 2m \ln(r - 2m) + Z(\tilde{t}), & \text{if } f = 1, \\ Z(\tilde{t})(r - 2m)^{2m} e^r, & \text{if } f = z, \\ Z(\tilde{t}) + r, & \text{if } f = \left(1 - \frac{2m}{r}\right), \end{cases} \quad (3.9)$$

where  $Z(\tilde{t})$  is an arbitrary function of  $\tilde{t}$ , we can set  $Z = z_0$  according to Lemma 1. In fact, we can choose any given monotone increasing function  $z(r)$  in this case, correspondingly we have

$$f = \left(1 - \frac{2m}{r}\right) z'(r). \quad (3.10)$$

So the option of  $f > 0$  is quite arbitrary.

In the case of the metric generated by rotating source similar to the Kerr ones[7, 8], we can not construct a null vector field  $V^\mu$  satisfying the integrable 1-form (2.6), so the corresponding metric can not be generally converted into the canonical form. Now we examine the rotating case. Generally the metric with rotation takes the following form in the coordinate system  $(t, r, \theta, \phi)$ ,

$$g_{\mu\nu} = \begin{pmatrix} u^2 & 0 & 0 & uw \\ 0 & -a & 0 & 0 \\ 0 & 0 & -b & 0 \\ uw & 0 & 0 & w^2 - v \end{pmatrix}, \quad (3.11)$$

where  $u, v, w, a, b$  are smooth functions of  $(r, \theta)$ , but independent of  $(t, \phi)$ . For the 4-vector speed

$$V^\mu = (\dot{t}, \dot{r}, \dot{\theta}, \dot{\phi}), \quad (3.12)$$

after some arrangement, the geodesic equation

$$\dot{V}^\alpha = -\Gamma_{\mu\nu}^\alpha V^\mu V^\nu \quad (3.13)$$

becomes

$$\begin{aligned} \frac{d}{d\tau} \dot{t} &= -\frac{1}{uv} \left( (2v - w^2) \frac{du}{d\tau} + uw \frac{dw}{d\tau} \right) \dot{t} \\ &\quad - \frac{1}{u^2 v} \left( w(v - w^2) \frac{du}{d\tau} + u(v + w^2) \frac{dw}{d\tau} - uw \frac{dv}{d\tau} \right) \dot{\phi}, \end{aligned} \quad (3.14)$$

$$\frac{d}{d\tau} \dot{\phi} = -\frac{1}{v} \left( w \frac{du}{d\tau} - u \frac{dw}{d\tau} \right) \dot{t} - \frac{1}{uv} \left( w^2 \frac{du}{d\tau} - uw \frac{dw}{d\tau} + u \frac{dv}{d\tau} \right) \dot{\phi}, \quad (3.15)$$

$$\begin{aligned} \frac{d}{d\tau} (a \dot{r}^2) &= - \left( 2u \partial_r u \dot{t}^2 + (2w \partial_r u + 2u \partial_r w) \dot{t} \dot{\phi} + (2w \partial_r w - \partial_r v) \dot{\phi}^2 \right) \dot{r} \\ &\quad + \partial_r b \dot{r} \dot{\theta}^2 - \partial_\theta a \dot{r}^2 \dot{\theta}, \end{aligned} \quad (3.16)$$

$$\begin{aligned} \frac{d}{d\tau} (b \dot{\theta}^2) &= - \left( 2u \partial_\theta u \dot{t}^2 + (2w \partial_\theta u + 2u \partial_\theta w) \dot{t} \dot{\phi} + (2w \partial_\theta w - \partial_\theta v) \dot{\phi}^2 \right) \dot{\theta} \\ &\quad - \partial_r b \dot{r} \dot{\theta}^2 + \partial_\theta a \dot{r}^2 \dot{\theta}. \end{aligned} \quad (3.17)$$

(3.14) and (3.15) are integrable due to the two Killing vectors  $(\partial_t, \partial_\phi)$ . The first integrals of (3.14) and (3.15) are given by

$$\dot{t} = -m \frac{w^2 - v}{vu^2} - n \frac{w}{uv}, \quad \dot{\phi} = m \frac{w}{uv} + n \frac{1}{v}, \quad (3.18)$$



where  $m, n$  are constants. Substituting (3.18) into the line element equation, we have

$$a\dot{r}^2 + b\dot{\theta}^2 = \frac{m^2}{u^2} - \frac{(nu + mw)^2}{u^2v} - C, \quad (3.19)$$

where  $C = 0$  for null geodesics, and  $C = 1$  for time-like geodesics.

By (3.11) and (3.18), the covariant speed becomes

$$V_\mu = g_{\mu\nu}V^\nu = (m, -a\dot{r}, -b\dot{\theta}, -n). \quad (3.20)$$

$V_t$  and  $V_\phi$  are constants, which correspond to the conserved Nöther charges, namely the energy  $E = p_t$  and angular momentum  $J_3 = p_\phi$  related with the Killing vectors  $(\partial_t, \partial_\phi)$ .

According to Remark 4, the metric can be converted into the canonical form if and only if there exists a function  $T(t, r, \theta, \phi)$  such that the null vector  $V_\mu = \partial_\mu T$ . Then by (3.20), we have

$$T = mt - k\theta - n\phi - h(r, \theta), \quad (3.21)$$

where  $k$  is a constant, and

$$\dot{r} = \frac{1}{a}\partial_r h, \quad \dot{\theta} = \frac{1}{b}(k + \partial_\theta h). \quad (3.22)$$

By (3.21) we find  $m$  is the scale of time, so we set  $m = 1$ . Substituting (3.22) into (3.16) and (3.17), we get

$$(\partial_r h)^2 = a \left( \frac{1}{u^2} - \frac{(nu + w)^2}{u^2v} - \frac{k^2}{b} \right), \quad \partial_\theta h = 0 \quad (3.23)$$

and

$$k \in \{0, 1\}. \quad (3.24)$$

By (3.23), we find  $h = h(r)$ .

In (3.24),  $k = 0$  corresponds to the case that  $\theta$  is the latitude in spherical coordinate system, but  $k = 1$  corresponds to the cases that  $\theta$  is the radius of spherical coordinate system or coordinates in the system similar to cylindrical coordinate system and so on. In the present case, we have  $k = 0$ .

By (3.23) and  $k = 0$ , we get

$$a = \frac{u^2 v h'(r)^2}{v - (nu + w)^2}. \quad (3.25)$$

(3.25) is the necessary and sufficient condition that the metric (3.11) can be converted into the canonical form (2.1).

Comparing the Kerr metric in the Boyer-Lindquist form with (3.11)[2, 4, 7, 8], we obtain

$$u^2 = \frac{r^2 + \alpha^2 \cos^2 \theta - 2mr}{r^2 + \alpha^2 \cos^2 \theta}, \quad (3.26)$$

$$v = \frac{2\alpha^2 mr(r^2 + \alpha^2 \cos^2 \theta + 6mr) \sin^4 \theta}{(r^2 + \alpha^2 \cos^2 \theta - 2mr)(r^2 + \alpha^2 \cos^2 \theta)} + (r^2 + \alpha^2) \sin^2 \theta, \quad (3.27)$$

$$w = \frac{4\alpha mr \sin^2 \theta}{\sqrt{(r^2 + \alpha^2 \cos^2 \theta - 2mr)(r^2 + \alpha^2 \cos^2 \theta)}}, \quad (3.28)$$

$$a = \frac{r^2 + \alpha^2 \cos^2 \theta}{r^2 - 2mr + \alpha^2}, \quad b = r^2 + \alpha^2 \cos^2 \theta, \quad (3.29)$$

where  $m$  is the mass of a star, and  $\alpha$  is a constant proportional to the angular momentum. Substituting (3.26)~(3.29) into (3.25), we find it contradicts  $\partial_\theta h = 0$ , so the Kerr metric can not be converted into the canonical form. Or equivalently, we can not construct a global light-cone coordinate system in the Kerr's space-time.

Now we transform the metric (3.11) with (3.25) into the canonical form. For (3.25), we make transformation  $\tilde{r} = h(r)$ , then we remove the function  $h(r)$  from the metric in the new system  $(t, \tilde{r}, \theta, \phi)$ . This procedure is equivalent to setting  $h(r) = r$ .

Substituting  $h = r$  and  $k = 0$  into (3.21), we get the new time coordinate  $\tilde{t}$

$$\tilde{t} = t - n\phi - r. \quad (3.30)$$

In (3.30), the term  $t - n\phi$  means a rigidly rotating movement of the coordinate system with constant angular speed  $\frac{1}{n}$  in the direction  $\partial_\phi$ . The total effect caused by  $n$  is to transform  $w$  in metric (3.11) and Eq.(3.25) into  $w + nu$ , so not losing generality, we set

$$n = 0, \quad \tilde{t} = t - r. \quad (3.31)$$

By (3.22) and (3.24), the covariant speed  $V^\mu$  defined in (3.12) becomes

$$V^\mu = \left( \frac{v - w^2}{u^2 v}, \frac{v - w^2}{u^2 v}, 0, \frac{w}{uv} \right). \quad (3.32)$$

Resolving (2.13) with  $f = V^r$ , we obtain the new coordinate  $z$

$$z = r + Z(\tilde{t}, \theta, \Phi), \quad (3.33)$$

where  $Z$  is an arbitrary function with

$$\Phi = \phi - \int \frac{uw}{v - w^2} dr. \quad (3.34)$$

We take  $Z = 0$  for simplicity.

Resolving (2.23), we get

$$F = F(\tilde{t}, \theta, \Phi). \quad (3.35)$$

we can choose any two independent functions

$$x = X(\tilde{t}, \theta, \Phi), \quad y = Y(\tilde{t}, \theta, \Phi), \quad (3.36)$$

as the new coordinates. In the new coordinate system  $(\tilde{t}, z, x, y)$  defined by (3.31), (3.33) and (3.36), the metric (3.11) becomes

$$(\tilde{g}_{\mu\nu}) = J^*(g_{\alpha\beta})J^{-1}, \quad (3.37)$$

in which  $J$  is the Jacobian matrix

$$J \stackrel{\text{def}}{=} \frac{\partial(\tilde{t}, z, x, y)}{\partial(t, r, \theta, \phi)}, \quad J^* \stackrel{\text{def}}{=} (J^{-1})^+. \quad (3.38)$$

The calculations show that we always have

$$\tilde{g}_{zz} = \tilde{g}_{zx} = \tilde{g}_{zy} = 0, \quad (3.39)$$

which means that, the  $z$  lines are always orthogonal to the surface  $(x, y)$ .

However  $\tilde{g}_{xy} \neq 0$  for arbitrarily chosen functions (3.36). This is natural. We should make a further coordinate transformation between  $(x, y)$  to set them orthogonal to each other. For instance, in the case of

$$\frac{uw}{v - w^2} = \frac{d}{dr}\varphi(r), \quad (3.40)$$

where  $\varphi(r)$  is any smooth function independent of  $\theta$ , the result is simple. We have the new coordinate

$$(\tilde{t}, z, x, y) = (t - r, r, \theta, \phi - \varphi), \quad (3.41)$$

and the corresponding new metric

$$\tilde{g}_{\mu\nu} = \begin{pmatrix} u^2 & a & 0 & uw \\ a & 0 & 0 & 0 \\ 0 & 0 & -b & 0 \\ uw & 0 & 0 & w^2 - v \end{pmatrix}. \quad (3.42)$$

#### IV. CONCLUSION

The above discussion shows that we can establish a kind of light-cone coordinate system for the space-time with a null gradient field. In such coordinate system, the metric takes a wonderful canonical form (2.1), and the Einstein's field equation becomes quite simple[5, 6]. This coordinate system might be also helpful to understand the propagation of the gravitational wave.

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